BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 12

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Contents

I. Review

Last time we:

(1) Discussed the correspondence between morphisms of Riemann surfaces and extensions of function fields. In particular, we saw that morphisms of degree d correspond to extensions of degree d.
(2) Showed that Galois morphisms correspond to Galois function field extensions, and noted that, in this case, the group of deck transformations is isomorphic to the Galois group of the function field extension.

II. Uniformization of Riemann surfaces

II.1. Universal covers of Riemann surfaces.

Theorem 1. *Every simply connected Riemann surface is isomorphic to exactly one of the following: (i) $\mathbb{P}^1$, (ii) $\mathbb{C}$, or (iii) $\mathfrak{D}$.*

This means that if X is a Riemann surface, its universal cover $\widetilde{X}$ must be one of these three. Moreover, letting $G = \mathrm{Deck}(\widetilde{X}/X)$, then $X \cong G\backslash\widetilde{X}$, so every Riemann surface can be expressed as a quotient of either the Riemann sphere, the complex plane, or the unit disc.

Theorem 2 (Uniformization of compact, connected Riemann surfaces)**.** *According to their universal coverings, compact, connected Riemann surfaces can be classified as follows:*

- $\mathbb{P}^1$ *is the only compact Riemann surface of genus* 0*.*
- *Every compact, connected Riemann surface of genus* 1 *is isomorphic to* $\mathbb{C}/\Lambda$ *for some full lattice* $\Lambda \leq \mathbb{C}$*.*

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- *Every compact, connected Riemann surface of genus ≥ 2 is isomorphic to a quotient $\Gamma\backslash\mathfrak{H}$ for some subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ acting freely and properly discontinuously on $\mathfrak{H}$.*

II.2. $\mathrm{PSL}_2(\mathbb{R})$ **as the group of isometries of hyperbolic space.**

Definition 3. The hyperbolic metric on the upper half-plane $\mathfrak{H}$ with coordinate $z = x + iy$ is defined by

$$\frac{|dz|^2}{(\mathrm{Im}(z))^2} := \frac{(dx)^2 + (dy)^2}{y^2}\,.$$

The notation $|dz|^2$ is used for the Euclidean metric $(dx)^2 + (dy)^2$ because it hints at its transformation property: transforming $|dz|^2$ by a holomorphic map f results in $|f'(z)|^2|dz|^2$.

One can prove this using the Cauchy-Riemann equations. Recall that if $f = u + iv$, then

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where $f'(z_0) = a + bi$, so $u_x = v_y$ and $v_x = -u_y$. Then

$$\begin{aligned} |df|^2 &= (du)^2 + (dv)^2 = (u_x\,dx + u_y\,dy)^2 + (v_x\,dx + v_y\,dy)^2 \\ &= u_x^2(dx)^2 + 2u_xu_y\,dx\,dy + u_y^2(dy)^2 + v_x^2(dx)^2 + 2v_xv_y\,dx\,dy + v_y^2(dy)^2 \\ &= u_x^2(dx)^2 - \cancel{2v_yv_x\,dx\,dy} + u_y^2(dy)^2 + v_x^2(dx)^2 + \cancel{2v_xv_y\,dx\,dy} + v_y^2(dy)^2 \\ &= (u_x^2 + v_x^2)(dx)^2 + (u_y^2 + v_y^2)(dy)^2 = |f'|^2((dx)^2 + (dy)^2) = |f'||dz|\,. \end{aligned}$$

The whole point of a Riemannian metric is that it allows us to endow a Riemannian manifold with the structure of a metric space by defining a notion of length that respects the smooth structure.

Definition 4.

- Let $\gamma : [a, b] \to \mathfrak{H}$ be a smooth curve and write $\gamma(t) = (x(t), y(t))$. The length of γ is
$$\ell(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,dt$$
- The area of a set $A \subseteq \mathfrak{H}$ is
$$a(A) = \int \frac{dx\,dy}{y^2}\,.$$

Definition 5.

- Define the hyperbolic distance $d_h : \mathfrak{H} \times \mathfrak{H} \to \mathbb{R}_{\geq 0}$ by
$$d_h(z, w) := \inf\{\ell(\gamma) : \gamma \text{ is a path from } z \text{ to } w\}\,.$$
- A path attaining this infimum, i.e., realizing the distance between two points, is called a geodesic.
- An open hyperbolic disc is a set of the form
$$D_h(z_0, r) = \{z \in \mathfrak{H} : d_h(z, z_0) < r\}$$
for some $z_0 \in \mathfrak{H}$ and $r \in \mathbb{R}_{\geq 0}$.

Proposition 6. *The set of orientation-preserving isometries of $\mathfrak{H}$ with respect to the hyperbolic metric is exactly* $\mathrm{PSL}_2(\mathbb{R})$.

Proof. We'll show one direction, namely that elements of $\mathrm{PSL}_2(\mathbb{R})$ are isometries. Given $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$, so $T(z) = \dfrac{az+b}{cz+d}$, one can compute

$$T'(z) = \frac{1}{(cz+d)^2} \quad \text{and} \quad \mathrm{Im}(T(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2}\,.$$

Then

$$\frac{|dT(z)|^2}{\mathrm{Im}(T(z))^2} = \frac{|T'(z)|^2|dz|^2}{\mathrm{Im}(T(z))^2} = \frac{\frac{1}{|cz+d|^4}|dz|^2}{\frac{\mathrm{Im}(z)^2}{|cz+d|^4}} = \frac{|dz|^2}{(\mathrm{Im}(z))^2}\,,$$

as desired. □

Recall that the map

$$\mathfrak{H} \to \mathfrak{D}$$
$$z \mapsto \frac{z-i}{z+i}$$

is an isomorphism of Riemann surfaces. One can show that the corresponding hyperbolic metric on $\mathfrak{D}$ under this isomorphism is given by

$$4\frac{|dz|^2}{(1-|z|^2)^2} = 4\frac{(dx)^2+(dy)^2}{(1-x^2-y^2)^2}\,.$$

Define a generalized circle in $\mathbb{C}$ (or $\widehat{\mathbb{C}}$) to be either a usual circle, or a straight line. Recall from complex analysis that generalized circles are preserved by Möbius transformations, i.e., elements of $\mathrm{PSL}_2(\mathbb{C})$. One can show this by first showing that all generalized circles can be expressed in the form

$$Az\bar{z} + Bz + \overline{B}\bar{z} + C = 0$$

where $A, C \in \mathbb{R}$ and $B \in \mathbb{C}$, and then showing that equations of this form are preserved by Möbius transformations.

We now determine the geodesics in $\mathfrak{H}$. As a special case, first assume that the points $z = ip$ and $w = iq$ both lie on the imaginary axis, and further assume $p < q$. Given a smooth path $\gamma(t) = (x(t), y(t))$, $t \in [0,1]$ from z to w, then

$$\begin{aligned}\ell(\gamma) &= \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,dt \geq \int_0^1 \frac{y'(t)}{y(t)}\,dt = \log(y(1)) - \log(y(0)) \\ &= \log(iq) - \log(ip) = \log(q/p)\,.\end{aligned}$$

Moreover, we have equality iff $x(t) = 0$ and $y'(t) \geq 0$ for all $t \in [0,1]$. For instance, $\gamma(t) = (0, (1-t)p + tq)$ is such a path. Thus $d_h(ip, iq) = \log(q/p)$, and the straight line segment between them along the imaginary axis is a geodesic.

Given arbitrary points $z, w \in \mathfrak{H}$, we can choose an element $T \in \mathrm{PSL}_2(\mathbb{R})$ such that $T(z) = i$. Moreover, after composing with a suitable hyperbolic rotation around i, we can assume that $T(w)$ lies on the imaginary axis. As we computed above, the geodesic between $T(z)$ and $T(w)$ is the vertical segment along the imaginary axis, so the geodesic between z and w is the image of this segment under T^{-1}.

Thus every geodesic in $\mathfrak{H}$ is the image of the imaginary axis under some element of $\mathrm{PSL}_2(\mathbb{R})$. Since Möbius transformations preserve generalized circles, these must be generalized circles, and one can show that those arising from elements of $\mathrm{PSL}_2(\mathbb{R})$ are orthogonal to $\partial\mathfrak{H} = \mathbb{R}$. Conversely, if C is a generalized circle orthogonal to $\mathbb{R}$ and $z_1, z_2 \in C$, then C is the geodesic between them.

Here is a summary of relevant facts about geodesics in $\mathfrak{H}$.

- The geodesics of $\mathfrak{H}$ are exactly the generalized circles orthogonal to $\partial\mathfrak{H} = \mathbb{R}$.
- $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on the set of all geodesics, i.e., given any two geodesics, there exists an element of $\mathrm{PSL}_2(\mathbb{R})$ sending one to the other.
- Two points $z, w \in \mathfrak{H}$ determine a unique geodesic.

[Show image at `https://upload.wikimedia.org/wikipedia/commons/a/a8/ModularGroup-Fundamen` `svg`.]

Remark 7. All of these facts have analogues for $\mathfrak{D}$. The real interval $(-1, 1)$ is a geodesic of $\mathfrak{D}$, and every geodesic is a generalized circle that is orthogonal to the unit circle.

[Show image at `https://en.wikipedia.org/wiki/File:Hyperbolic_domains_642.png`.]

While we have given a procedure for computing the hyperbolic distance between two points in $\mathfrak{H}$, one can even give an explicit formula:

$$d_h(z_1, z_2) = 2\operatorname{arctanh}\left(\left|\frac{z_1 - z_2}{z_1 - \overline{z_2}}\right|\right)$$

or equivalently,

$$d_h(x_1 + iy_1, x_2 + iy_2) = \operatorname{arccosh}\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1y_2}\right).$$

III. Fuchsian groups

Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$. Recall that Γ acts properly discontinuously if for each $z \in \mathfrak{H}$ there exists an open neighborhood $U \ni z$ such that there are only finitely many $\gamma \in \Gamma$ such that $\gamma(U) \cap U \neq \varnothing$.

Also recall that:

- Γ acts properly discontinuously $\implies$ $\Gamma\backslash\mathfrak{H}$ is Hausdorff;
- Γ acts freely and properly discontinuously $\implies$ the quotient map $\pi : \mathfrak{H} \to \Gamma\backslash\mathfrak{H}$ is a covering map with covering group Γ.

In general, our quotient maps may be ramified, so it's too restrictive to insist that Γ act freely. But we at least want our quotients to be Hausdorff.

Definition 8. A Fuchsian group is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, i.e., a subgroup such that the subgroup topology is the discrete topology.

Remark 9. Recall that Γ is discrete iff for each $\gamma \in \Gamma$ there is an open subset $U \subseteq \mathrm{PSL}_2(\mathbb{R})$ with $\gamma \in U$ such that $U \cap \Gamma = \{\gamma\}$.

Lemma 10. *Let Γ be a Fuchsian group. For every $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ (not necessarily in Γ) there exists a neighborhood $V \subseteq \mathrm{PSL}_2(\mathbb{R})$ of α such that $V \cap \Gamma$ is finite. In particular, Γ is a closed subset of $\mathrm{PSL}_2(\mathbb{R})$.*

Proof. See Lemma 2.18 of Girondo–González-Diez. □

Proposition 11. *A subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ is Fuchsian iff it acts properly discontinuously on $\mathfrak{H}$.*

Proof. ($\Leftarrow$): We prove the contrapositive; assume Γ is not Fuchsian. Then there exists $\gamma \in \Gamma$ and an infinite sequence of distinct elements $\gamma_n \in \Gamma$ such that $\lim_{n\to\infty} \gamma_n = \gamma$. Setting $\beta_n = \gamma^{-1}\gamma_n$, then $\lim_{n\to\infty} \beta_n = 1$. Thus for any $z \in \mathfrak{H}$ we have $\beta_n(z) \to z$ as $n \to \infty$. But then for any neighborhood $U \ni z$ there are infinitely many $\beta_n \in \Gamma$ such that $\beta_n(U) \cap U \neq \varnothing$, so Γ doesn't act properly discontinuously.

($\Rightarrow$): Assume Γ is Fuchsian, and for contradiction, assume that Γ doesn't act properly continuously. Then there exists $z_0 \in \mathfrak{H}$ and an infinite sequence (γ_k) of distinct elements of Γ such that $\gamma_k(U_k) \cap U_k \neq \varnothing$, where $U_k = D_h(z_0, 1/k)$, the hyperbolic disc of radius $1/k$. Then the sequence (γ_k) is contained in the set

$$C := \{\alpha \in \mathrm{PSL}_2(\mathbb{R}) : d_h(z_0, \alpha(z_0)) \leq 1\}$$

which one can show is compact. Thus (γ_k) is contained in $\Gamma \cap C$. By the previous lemma, $\Gamma \cap C$ is a closed subset of C, hence it is compact. Then $\Gamma \cap C$ is both discrete and compact, hence must be finite. Contradiction. □

Proposition 12. *If Γ is a Fuchsian group, then the quotient $\Gamma\backslash\mathfrak{H}$ can be equipped with the structure of a Riemann surface such that the quotient map $\pi : \mathfrak{H} \to \Gamma\backslash\mathfrak{H}$ is a morphism of Riemann surfaces.*

Proof sketch. By the above, Γ acts properly discontinuously, so $\Gamma\backslash\mathfrak{H}$ is Hausdorff by previous results. If $\mathrm{Stab}_\Gamma(z) = \{1\}$, then the quotient map π is a covering map, hence a local homeomorphism, on a neighborhood of z. Thus we can simply use the corresponding complex chart on $\mathfrak{H}$ to define a chart containing $[z]$.

If instead $z \in \mathfrak{H}$ is fixed by some element of Γ, we can find a sufficiently small neighborhood U of z such that π locally looks like $z \mapsto z^n$. That is, such that we have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[d, "\psi_1"'] & \Gamma\backslash U \arrow[d, "\psi_2"] \\ \mathfrak{D} \arrow[r] & \mathfrak{D} \\ z \arrow[r, mapsto] & z^n \end{tikzcd}$$

where ψ_1 and ψ_2 are isomorphisms of Riemann surfaces. Then we can simply take (U, ψ_2) as a chart containing $[z]$, and one can show that such charts are holomorphically compatible. □

Proposition 13. *A Fuchsian group Γ acts freely on $\mathfrak{H}$ iff Γ is torsion-free.*

Proof idea. Torsion elements are exactly those that are conjugate to rotations. □

Proposition 14. *Let X_1 and X_2 be Riemann surfaces uniformized by Fuchsian groups Γ_1 and Γ_2 acting freely on $\mathfrak{H}$, so $X_1 \cong \Gamma_1\backslash\mathfrak{H}$ and $X_2 \cong \Gamma_2\backslash\mathfrak{H}$. Then $X_1 \cong X_2$ iff Γ_1 and Γ_2 are conjugate in* $\mathrm{PSL}_2(\mathbb{R})$, *i.e., there exists $T \in \mathrm{PSL}_2(\mathbb{R})$ such that $T\,\Gamma_1 T^{-1} = \Gamma_2$.*

Proof. ($\Leftarrow$): We claim that the isomorphism $T : \mathfrak{H} \to \mathfrak{H}$ induces an isomorphism

$$\begin{aligned} \Gamma_1\backslash\mathfrak{H} &\xrightarrow{\sim} \Gamma_2\backslash\mathfrak{H} \\ [z]_1 &\mapsto [T(z)]_2 \,. \end{aligned}$$

To show that this is well-defined, suppose that $[z]_1 = [w]_1$, so $w = \gamma_1(z)$ for some $\gamma_1 \in \Gamma_1$. Then $T \circ \gamma_1 \in T\,\Gamma_1 = \Gamma_2 T$, so $T \circ \gamma_1 = \gamma_2 \circ T$ for some $\gamma_2 \in \Gamma_2$. Then

$$[T(w)]_2 = [T(\gamma_1(z))]_2 = [\gamma_2(T(z))]_2 = [T(z)]_2$$

so the map is well-defined.

Conversely, if $\varphi : \Gamma_1\backslash\mathfrak{H} \xrightarrow{\sim} \Gamma_2\backslash\mathfrak{H}$ is an isomorphism, then there is a lift $T : \mathfrak{H} \to \mathfrak{H}$ such that the following diagram commutes. (This actually requires some results on covering spaces and lifts.)

$$\begin{tikzcd} \mathfrak{H} \arrow[r, "T"] \arrow[d, "p_1"] & \mathfrak{H} \arrow[d, "p_2"] \\ \Gamma_1\backslash\mathfrak{H} \arrow[r, "\varphi"'] & \Gamma_2\backslash\mathfrak{H} \end{tikzcd}$$

One can then show that $T\,\Gamma_1 T^{-1} = \Gamma_2$. $\square$

Definition 15. Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$, and let D be a simply connected closed subset of $\mathfrak{H}$ whose boundary ∂D consists of a finite union of differentiable paths. D is a fundamental domain for Γ if $\{\gamma(D) : \gamma \in \Gamma\}$ tessellates $\mathfrak{H}$, i.e,

(1) $\bigcup_{\gamma\in\Gamma} \gamma(D) = \mathfrak{H}$; and
(2) for every $\gamma \in \Gamma \setminus \{1\}$, the intersection $D \cap \gamma(D)$ is contained in the boundary of D.

Remark 16. In other words, any two translates of D don't intersect, except possibly on their boundaries. More formally,

$$D^\circ \cap (\gamma D)^\circ = \varnothing$$

for all $1 \neq \gamma \in \Gamma$, where D° denotes the interior of D.

[Show picture of fundamental domain of modular group: `https://en.wikipedia.org/wiki/File:ModularGroup-FundamentalDomain.svg`. Also show picture on p. 634 of Voight's quaternion algebras book.]

Remark 17. Fundamental domains are very useful for obtaining a concrete view of a group G acting on a space X.

(1) They provide an (almost) unique representative of each equivalence class $[x] \in G\backslash X$.
(2) They show how the quotient space $G\backslash X$ is glued together. [Show image at `https://beta.lmfdb.org/Belyi/6T15/5.1/5.1/4.2/a`.]

IV. Fuchsian triangle groups

A hyperbolic triangle in $\mathfrak{H}$ is a topological triangle whose edges are hyperbolic geodesic segments. We allow the possibility of triangles with edges of infinite length, in which case at least one of the vertices lies in $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

The characteristic property of hyperbolic spaces is the fact that the sum of the angles of a hyperbolic triangle is less than π.

Proposition 18. *If T is a hyperbolic triangle with angles α, β, γ, then the hyperbolic area of T is $a(T) = \pi - \alpha - \beta - \gamma$.*